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ON THE DERIVATION OF WELL POSED BOUNDARY VALUE PROBLEMS IN STRUCTURAL MECHANICS†

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Abstract—The paper starts with a discussion of a number of published boundary value problems in structural mechanics, whose intuitive formulation led to incorrect boundary or matching conditions. It is then shown on three examples, using variational methods, how to obtain well posed formulations for problems of this type. For uniformity of presentation, all examples deal with continuously supported beams. The first two examples exhibit boundary and matching points whose position is *fixed* along the beam axis. In the third example it is shown how to use the method of variational calculus for a variable end point to formulate structural problems with *variable* matching points. This is demonstrated on the problem of a beam which rests on, but is not attached to, a Pasternak base; thus where parts of the beam may lift off the base. Each example concludes with a comparison and discussion of related formulations published in the literature.

INTRODUCTION

Problems in structural mechanics which involve beams, plates, and shells are usually formulated in terms of differential equations, boundary conditions, and initial conditions.§

In one approach, the DE's for a problem under consideration are derived by setting up equations of motion, or equilibrium, on an infinitesimal element of the body and then by performing the limit of shrinking the size of the element to a point. The corresponding BC's are stated so as to satisfy the apparent geometrical and mechanical conditions at the boundaries.

In connection with this approach a number of questions arise. One question is whether a stated set of BC's, which reflects the investigator's view of the conditions at the boundary, is correct mechanically. This leads to another related question whether the resulting formulation is mathematically well posed, (i.e. whether the formulation is mathematically consistent).

For standard problems, such as beams or plates simply supported or fixed along the boundary, the formulation of the corresponding BC's is simple. It leads, mechanically and mathematically, to well posed problems.

For some other structural problems this intuitive approach may encounter difficulties. An early well-known example of such a situation is the difficulty encountered by Poisson[1], when formulating the BC's along a free edge for the bending theory of thin plates. Based on his knowledge of the theory of elasticity and his view of the mechanical conditions at the free boundary, Poisson prescribed three conditions at each point of the boundary, whereas the fourth order elliptic partial differential equation can accept only two conditions.

Other examples, of more recent origin, occurred when formulating the BC's for a "free" edge of continuously supported structures. For beams, plates, and shells on a Winkler base, the BC's are not affected by the base. However, when other foundation models are used (for example those by Wieghardt, Pasternak, etc.), because of the simplifying assumptions made, concentrated reaction forces may occur along the free edges and they have to be included in the formulation of the BC's. The respective difficulties encountered by Wieghardt[2] and Pflanz[3] were pointed out and clarified by Kerr[4]. Errors of the same type committed more recently by other investigators (for example, Ref. [5]) will be discussed later.

Another class of problems for which the intuitive approach may lead to difficulties, is the formulation of the matching conditions between two regions that are governed by different DE's.

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§In the following, for the sake of brevity and easier identification, a differential equation is denoted by DE and a boundary condition by BC.

The difficulties are compounded when the matching point, or line, is not fixed in space. An early example which exhibits these difficulties, is the analytical formulation for the determination of the buckling pressure of a spherical shell by Friedrichs[6]. An inconsistency in this formulation was pointed out by Mushtari and Galimov[7]. Other problems of this type published recently are the lift off problems of beams or plates from a continuous base (an example is shown in Fig. 4).

A second approach for formulating the differential equations and boundary conditions for problems in structural mechanics, utilizes a variational principle and the methods of variational calculus. For problems with fixed boundaries, this method is well established and widely used[8, 9]. This approach has the great advantage that for a given energy functional it generates the necessary DE's and a variety of admissible BC's from which a well posed formulation, mechanically and mathematically, may be chosen. This approach was used by Kirchhoff[10] for establishing the proper two BC's along the free edge of a thin plate. It was used since by others to formulate a variety of problems in structural mechanics[8, 9].

The purpose of the present paper is two-fold. In the first part (Examples I and II) using the variational approach, the proper boundary and matching conditions are derived for a number of problems with *fixed* matching points, which were incorrectly formulated in recent publications using the intuitive approach. It is then shown the cause of these errors and how to avoid them.

In the second part of the paper it is shown how to properly formulate structural problems with *variable* matching points. Several years ago, the author conceived the idea of using the method of variational calculus for variable end points to formulate structural problems with variable boundaries or *variable* matching points. This method yields in addition to the DE's and BC's also transversality conditions for locating the position of the variable matching points. At the suggestion of this writer, El-Bayoumy[11] utilized this approach for the solution of a confined ring problem. This method is presented in Example III. As part of this presentation it is shown that two lift-off problems for continuously supported beams and plates recently analyzed by Chernigovskaya[5] are incorrectly formulated.

EXAMPLE I: BEAM CONTINUOUSLY ATTACHED TO A PASTERNAK BASE

To study the formulation of proper matching conditions at a *fixed* boundary, consider a finite beam attached to a two-dimensional Pasternak base as shown in Fig. 1. To simplify the presentation it is assumed that the system is symmetrical with respect to the coordinate origin. Denoting by

$w_1(x)$ the deflections in $0 \leqq x \leqq l$ and by
$w_2(x)$ the deflections in $l \leqq x \leqq L$,

the total potential energy of the system becomes

$$\Pi = 2\int_0^l \left(\frac{EI}{2} w_1''^2 + \frac{G}{2} w_1'^2 + \frac{k}{2} w_1^2 - q w_1\right) dx + 2\int_l^L \left(\frac{G}{2} w_2'^2 + \frac{k}{2} w_2^2\right) dx - P w_1(0) \qquad (1.1)$$

where EI is the flexural rigidity of the beam, G is the parameter of the shear layer, and k is the parameter of the spring layer.

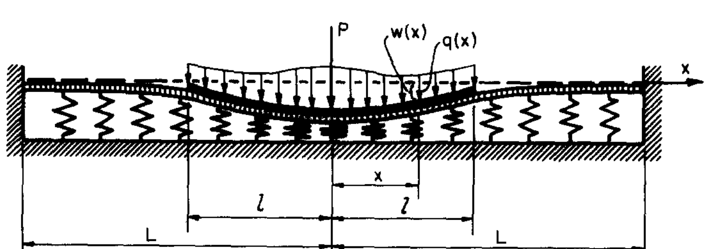

Fig. 1.

According to the principle of stationary total potential energy, $\delta\Pi = 0$. Performing the variations and then integrating by parts we obtain

$$\begin{aligned}\frac{1}{2}\delta\Pi = &\int_0^l [(EIw_1'')'' - (Gw_1')' + kw_1 - q]\delta w_1\, dx \\ &+ \int_l^L [-(Gw_2')' + kw_2]\delta w_2\, dx - \frac{P}{2}\delta w_1(0) + [EIw_1''\delta w_1']_0^l \\ &- \{[(EIw_1'')' - Gw_1']\delta w_1\}_0^l + [Gw_2'\delta w_2]_l^L = 0. \end{aligned} \tag{1.2}$$

Noting that (because of the fundamental lemma)

$$\begin{aligned} (EIw_1'')'' - (Gw_1')' + kw_1 = q \qquad & 0 \leqq x \leqq l \\ -(Gw_2')' + kw_2 = 0 \qquad & l \leqq x \leqq L \end{aligned} \tag{1.3}$$

and that because

$$w_1(l) = w_2(l) \tag{1.4}$$

the relation $\delta w_1(l) = \delta w_2(l)$ holds, eqn (1.2) reduces to

$$\begin{aligned}\frac{1}{2}\delta\Pi = &[EIw_1''\delta w_1']_0^l + \left\{\left[(EIw_1'')' - Gw_1' - \frac{P}{2}\right]\delta w_1\right\}_{x=0} \\ &- \{[(EIw_1'')' - G(w_1' - w_2')]\delta w_1\}_{x=l} + [Gw_2'\delta w_2]_{x=L} = 0. \end{aligned} \tag{1.5}$$

Thus, the BC's at $x = 0$ are

$$w_1'(0) = 0 \tag{1.6}$$

$$\left[(EIw_1'')' - Gw_1' - \frac{P}{2}\right]_{x=0} = 0 \tag{1.7}$$

the matching conditions at $x = l$ are

$$w_1(l) = w_2(l) \tag{1.4}$$

$$w_1''(l) = 0 \tag{1.8}$$

$$[(EIw_1'')' - G(w_1' - w_2')]_{x=l} = 0 \tag{1.9}$$

and the BC at $x = L$ is either

$$w_2(L) = 0 \tag{1.10}$$

when base adheres to the rigid surrounding (or a prescribed non-zero constant), or

$$w_2'(L) = 0 \tag{1.10'}$$

Note that in BC (1.7) the second term vanishes because of BC (1.6).

For a mechanical interpretation of the obtained boundary and matching conditions it should be noted that, as shown in [4], the shearing force in the Pasternak base is $S(x) = Gw'(x)$ and that, according to the bending theory, the shearing force in the beam is $V(x) = -(EIw'')'$. This is shown in Fig. 2.

Thus, condition (1.10′) states that the shearing force at $x = L$, i.e. between the base and the surrounding medium, is equal to zero. Also note that the second term in (1.9), which is due to a discontinuity in slope of the shear layer at $x = l$, represents a concentrated reaction. This concentrated reaction is an idealization which results from the simplifying assumptions made in formulating the base response. It represents in reality a strong increase of the reaction

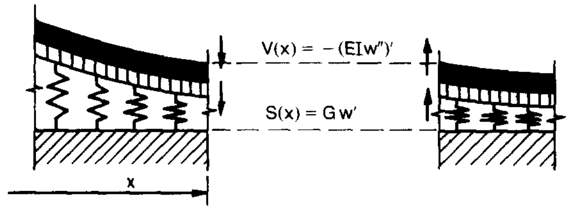

Fig. 2.

distribution in a narrow region near the edge. Wieghardt[2], treating a similar problem, assumed the occurence of concentrated reactions as physically impossible and so arrived at an overdetermined formulation. For a discussion of this question refer to [4]. More recently Chernigovskaya, analyzing a related problem, suggested ([5], p. 124) that the concentrated end reactions may be neglected "because of the plasticity of the soil". This is not an admissible practice, since the resulting formulation and solution will not satisfy vertical equilibrium.

EXAMPLE II: BEAM CONTINUOUSLY ATTACHED TO A "GENERALIZED BASE"

Another problem, closely related to the above, is the analysis of a continuously supported beam in which the base response is represented by a spring layer and a continuous reaction moment, as follows:

$$p(x) = k_1 w(x)$$
$$\mu(x) = k_2 w'(x) \tag{2.1}$$

In Ref. [4] this foundation model is referred to as the "generalized" foundation. There (on p. 496) it was shown that the resulting DE for a plate (or beam) attached to a "generalized" foundation is the same as the one for the plate (or beam) attached to a Pasternak foundation, if $G = k_2$. This observation suggests that the response of the "generalized" base and of the Pasternak base will be similar.

To demonstrate some of the difficulties encountered when formulating intuitively beam or plate problems attached to a "generalized" base consider, as an example, the beam problem shown in Fig. 3(a). For this system, the total potential energy is

$$\Pi = \int_0^l \left(\frac{EI}{2} w''^2 + \frac{k_2}{2} w'^2 + \frac{k_1}{2} w^2 - qw\right) dx. \tag{2.2}$$

The corresponding formulation obtained from $\delta\Pi = 0$ is, noting eqn (1.2), the DE

$$(EIw'')'' - (k_2 w')' + k_1 w = q \qquad 0 \leqq x \leqq l \tag{2.3}$$

and the BC's

$$w''(0) = 0; \quad w''(l) = 0 \tag{2.4}$$

$$[(EIw'')' - k_2 w']_{x=0} = 0; \quad [(EIw'')' - k_2 w']_{x=l} = 0. \tag{2.5}$$

Note that the second term in DE (2.3) and in BC (2.5) represent the contribution of the distributed base reaction moment. Thus, the distributed base reaction moments also affect a BC at the "free" end of a beam.

The continuous base reaction moments were recently introduced in the analyses for the determination of buckling temperatures of a railroad track by Pershin[12], Engel[13], and Prud'homme and Janin[14]. Assuming that the track buckles in the horizontal plane, the two rails were represented by one beam of constant cross section, the lateral resistance of the ballast by a spring layer (or Coulomb friction or a combination of both), and the torsional resistance of the closely spaced fasteners by a continuously distributed reaction moment that is proportional to the local angle of rotation[15]. Prud'homme and Janin included in their DE the term $(-k_2 w'')$ but stated the BC's for a "free" end of the track, at $x = l$, ([14], p. 603) as

$$w''(l) = 0; \quad w'''(l) = 0. \tag{2.6}$$

Comparing the second BC with those shown in (2.5), it follows that the second BC in (2.6) is not correct. To obtain a mathematically consistent formulation the term $(-k_2 w')$ has to be added (as well as the term due to the axial force if one exists).

For a physical interpretation of the term $(k_2 w')'$ in the DE (2.3) and the term $k_2 w'$ in the BC (2.5) consider a beam attached to a "generalized" foundation, as shown in Fig. 3(a). Obeying the positive sign convention, the corresponding reaction pressure and reaction moment distributions are shown in Fig. 3(b). Replacing the reaction moments by couples, as shown in Fig. 3(c), it follows that the distribution of reaction moments is statically equivalent to a vertical force distribution of intensity

$$\mu'(x) = (k_2 w')'$$

which enters the DE and concentrated reactions at the free ends of magnitude $k_2 w'$ which enter the BC's, as shown in Fig. 3(d).†

The foundation equations (2.1) were also used recently by Fletcher and Hermann [17] for the determination of the response of a semi-infinite beam embedded in an elastic continuum. Since for the analyzed problem the continuum extended beyond the free end of the beam, the formulation of the BC's at this end point requires special attention. For example, one approach could be based on the observed similarity of the "generalized" and Pasternak models, by assuming that the effect of the continuum beyond the free end of the beam may be expressed by a Pasternak base, as done in Example I. According to this approach, at the free end of the beam there will exist a concentration reaction force, but not a concentrated reaction moment as assumed in [17].

†This argument is analogous to the one used by Lord Kelvin and P. G. Tait [16] to explain the physical meaning of a BC for the free edge of a plate, originally derived by Kirchhoff [10] by means of the variational approach.

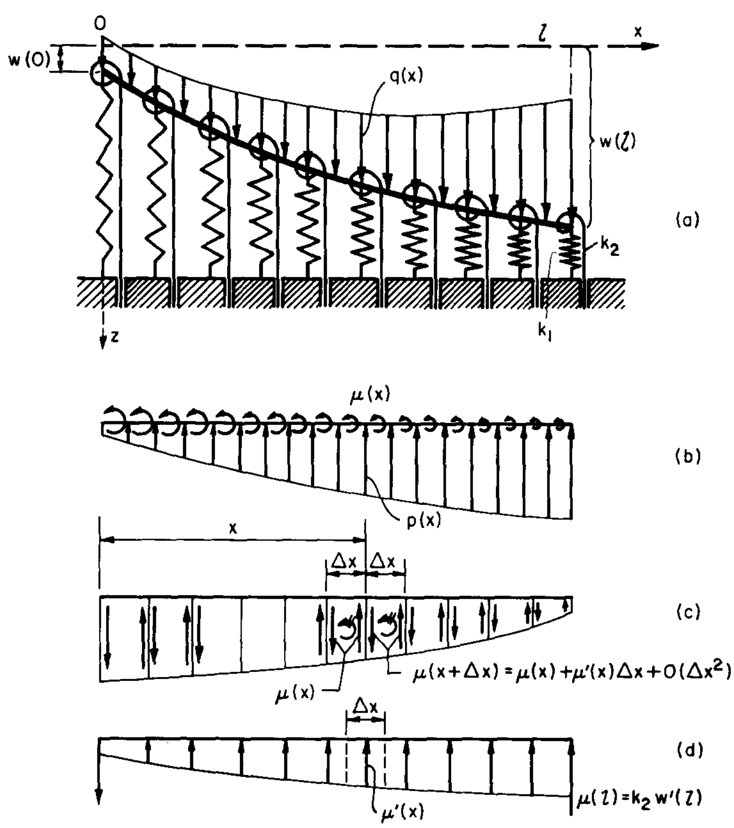

Fig. 3.

From the above discussion and derivations it appears that once the equations for the foundation and beam response are chosen, from the point of view of mathematical consistency, the concentrated reactions at a free end may not be prescribed arbitrarily. If they exist, they will appear in the corresponding variational formulation, as shown above.

EXAMPLE III: BEAM RESTING ON A PASTERNAK BASE (WITH LIFT-OFF)

To study the formulation of the conditions at a *variable* matching point, consider the beam-foundation problem analyzed in Example I, but assume that the beam is not attached to the Pasternak foundation. To simplify the presentation it is also assumed here that the system is symmetrical with respect to the origin.

Because the beam is not attached to the base, tensile stresses cannot occur between beam and base and therefore the beam may lift off the base over certain intervals, as shown in Fig. 4. The coordinate of the matching point, $x = a$, is not known a priori. It depends upon the mechanical and geometrical parameters of the problem. Thus, in addition to the usual boundary and matching conditions (that would be required for x = a fixed) an equation is needed for the determination of the unknown a.

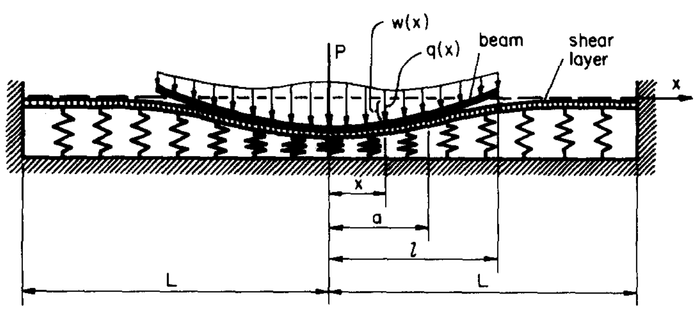

Fig. 4.

In the following, this additional equation is derived by utilizing the method of variational calculus for variable end points[18]. This method yields, in addition to the DE's and BC's, also a transversality condition at $x = a$, which is the needed equation. Denoting by

$w_1(x)$ the deflections of beam and base in $0 \leqq x \leqq a$
$w_2(x)$ the deflections of beam in the lift off region $a \leqq x \leqq l$
$w_s(x)$ the deflections of the base surface in $a \leqq x \leqq L$

the total potential energy of the system may be written as

$$\Pi = \int_0^a F_1[w_1(x), w_1'(x), w_1''(x)]\,dx + \int_a^l F_2[w_2(x), w_2''(x)]\,dx + \int_a^L F_s[w_s(x), w_s'(x)]\,dx - Pw_1(0) \tag{3.1}$$

where

$$\left.\begin{aligned} F_1 &= 2\left(\frac{EI}{2}w_1''^2 + \frac{G}{2}w_1'^2 + \frac{k}{2}w_1^2 - qw_1\right)\\ F_2 &= 2\left(\frac{EI}{2}w_2''^2 - qw_2\right)\\ F_s &= 2\left(\frac{G}{2}w_s'^2 + \frac{k}{2}w_s^2\right). \end{aligned}\right\} \tag{3.2}$$

To derive the first variation, $\delta\Pi$, we form the difference

$$\Delta\Pi = \left\{\int_0^{a+\delta a} F_1[w_1+\delta w_1, w_1'+\delta w_1', w_1''+\delta w_1'']\,\mathrm{d}x - \int_0^a F_1[w_1, w_1', w_1'']\,\mathrm{d}x\right\}$$
$$+\left\{\int_{a+\delta a}^{l} F_2[w_2+\delta w_2, w_2''+\delta w_2'']\,\mathrm{d}x - \int_a^l F_2[w_2, w_2'']\,\mathrm{d}x\right\}$$
$$+\left\{\int_{a+\delta a}^{L} F_s[w_s+\delta w_s, w_s'+\delta w_s']\,\mathrm{d}x - \int_a^L F_s[w_s, w_s']\,\mathrm{d}x\right\}$$
$$-P\{w_1(0)+\delta w_1(0)-w_1(0)\}. \tag{3.3}$$

Noting that the above expression may be rewritten as

$$\Delta\Pi = \left\{\int_0^a F_1[w_1+\delta w_1,\ldots]\,\mathrm{d}x - \int_0^a F_1[w_1,\ldots]\,\mathrm{d}x\right\} + \int_a^{a+\delta a} F_1[w_1,\ldots]\,\mathrm{d}x$$
$$+\left\{\int_a^l F_2[w_2+\delta w_2,\ldots]\,\mathrm{d}x - \int_a^l F_2[w_2,\ldots]\,\mathrm{d}x\right\} - \int_a^{a+\delta a} F_2[w_2,\ldots]\,\mathrm{d}x$$
$$+\left\{\int_a^L F_s[w_s+\delta w_s,\ldots]\,\mathrm{d}x - \int_a^L F_s[w_s,\ldots]\,\mathrm{d}x\right\} - \int_a^{a+\delta a} F_s[w_s,\ldots]\,\mathrm{d}x$$
$$-P\delta w_1(0) \tag{3.4}$$

it follows, using the mean value theorem on the integrals with the limits from a to $a+\delta a$, that

$$\delta\Pi = \delta\int_0^a F_1[w_1, w_1', w_1'']\,\mathrm{d}x + F_1[w_1, w_1', w_1'']|_{x=a}\delta a$$
$$+\delta\int_a^l F_2[w_2, w_2'']\,\mathrm{d}x - F_2[w_2, w_2'']|_{x=a}\delta a$$
$$+\delta\int_a^L F_s[w_s, w_s']\,\mathrm{d}x - F_s[w_s, w_s']|_{x=a}\delta a - P\delta w_1(0). \tag{3.5}$$

The second term in each row of eqn (3.5) is due to the fact that the matching point at $x = a$ is not fixed along the x axis.

The equilibrium equations, i.e. the formulation, is obtained from the condition

$$\delta\Pi = 0. \tag{3.6}$$

Performing the indicated variations in (3.5) and the usual integrations by parts, then noting the geometrical matching conditions at $x = a$

$$\left.\begin{array}{ll} w_1(a) = w_2(a) = w_s(a) \rightarrow & \delta w_1(a) = \delta w_2(a) = \delta w_s(a) \\ w_1'(a) = w_2'(a) & \rightarrow \delta w_1'(a) = \delta w_2'(a) \end{array}\right\} \tag{3.7}$$

and the fact that for the problem under consideration the Euler equations

$$\left.\begin{array}{ll} \dfrac{\partial F_1}{\partial w_1} - \left(\dfrac{\partial F_1}{\partial w_1'}\right)' + \left(\dfrac{\partial F_1}{\partial w_1''}\right)'' = 0 & 0 \leqq x \leqq a \\ \dfrac{\partial F_2}{\partial w_2} + \left(\dfrac{\partial F_2}{\partial w_2''}\right)'' = 0 & a \leqq x \leqq l \\ \dfrac{\partial F_s}{\partial w_s} - \left(\dfrac{\partial F_s}{\partial w_s'}\right)' = 0 & a \leqq x \leqq L \end{array}\right\} \tag{3.8}$$

have to be satisfied, condition (3.6) reduces to

$$\delta\Pi = [F_1 - F_2 - F_s]_{x=a}\delta a + \left\{\left[-\frac{\partial F_1}{\partial w_1'} + \left(\frac{\partial F_1}{\partial w_1''}\right)' - P\right]\delta w_1\right\}_{x=0} - \left\{\frac{\partial F_1}{\partial w_1''}\delta w_1'\right\}_{x=0}$$
$$+\left\{\left[\frac{\partial F_1}{\partial w_1'} - \left(\frac{\partial F_1}{\partial w_1''}\right)' + \left(\frac{\partial F_2}{\partial w_2''}\right)' - \frac{\partial F_s}{\partial w_s'}\right]\delta w_1\right\}_{x=a}$$

$$+\left\{\left(\frac{\partial F_1}{\partial w_1''}-\frac{\partial F_2}{\partial w_2''}\right)\delta w_1'\right\}_{x=a}+\left\{\frac{\partial F_2}{\partial w_2''}\delta w_2'\right\}_{x=l}$$

$$-\left\{\left(\frac{\partial F_2}{\partial w_2''}\right)'\delta w_2\right\}_{x=l}+\left\{\frac{\partial F_s}{\partial w_s'}\delta w_s\right\}_{x=L}=0. \tag{3.9}$$

Noting that according to Fig. 5

$$\delta w_1|_{x=a}=\delta w_a-w_1'(a)\delta a \tag{3.10}$$

and that

$$\delta w_1'|_{x=a}=\delta w_a'-w_1''(a)\delta a \tag{3.10'}$$

eqn (3.9) may be rewritten as follows:

$$\delta\Pi=\left\{\left[-\frac{\partial F_1}{\partial w_1'}+\left(\frac{\partial F_1}{\partial w_1''}\right)'-P\right]\delta w_1\right\}_{x=0}-\left\{\frac{\partial F_1}{\partial w_1''}\delta w_1'\right\}_{x=0}$$

$$+\left\{F_1-F_2-F_s-\left[\frac{\partial F_1}{\partial w_1'}-\left(\frac{\partial F_1}{\partial w_1''}\right)'+\left(\frac{\partial F_2}{\partial w_2''}\right)'-\frac{\partial F_s}{\partial w_s'}\right]w_1'-\left(\frac{\partial F_1}{\partial w_1''}-\frac{\partial F_2}{\partial w_2''}\right)w_1''\right\}_{x=a}\delta a$$

$$+\left\{\frac{\partial F_1}{\partial w_1'}-\left(\frac{\partial F_1}{\partial w_1''}\right)'+\left(\frac{\partial F_2}{\partial w_2''}\right)'-\frac{\partial F_s}{\partial w_s'}\right\}_{x=a}\delta w_a+\left\{\frac{\partial F_1}{\partial w_1''}-\frac{\partial F_2}{\partial w_2''}\right)_{x=a}\delta w_a'$$

$$+\left\{\frac{\partial F_2}{\partial w_2''}\delta w_2'\right\}_{x=l}-\left\{\left(\frac{\partial F_2}{\partial w_2''}\right)'\delta w_2\right\}_{x=l}+\left\{\frac{\partial F_s}{\partial w_s'}\delta w_s\right\}_{x=L}=0. \tag{3.11}$$

Since all the variations which appear in the above equation are independent, it follows that the BC's at $x=0$ are

$$w_1'(0)=0 \tag{3.12}$$

$$\left[-\frac{\partial F_1}{\partial w_1'}+\left(\frac{\partial F_1}{\partial w_1''}\right)'-P\right]_{x=0}=0 \tag{3.13}$$

the matching conditions at $x=a$ are

$$w_1(a)=w_2(a) \tag{3.14}$$

$$w_1(a)=w_s(a) \tag{3.15}$$

$$w_1'(a)=w_2'(a) \tag{3.16}$$

and

$$\left[\frac{\partial F_1}{\partial w_1''}-\frac{\partial F_2}{\partial w_2''}\right]_{x=a}=0 \tag{3.17}$$

$$\left[\frac{\partial F_1}{\partial w_1'}-\left(\frac{\partial F_1}{\partial w_1''}\right)'+\left(\frac{\partial F_2}{\partial w_2''}\right)'-\frac{\partial F_s}{\partial w_s'}\right]_{x=a}=0 \tag{3.18}$$

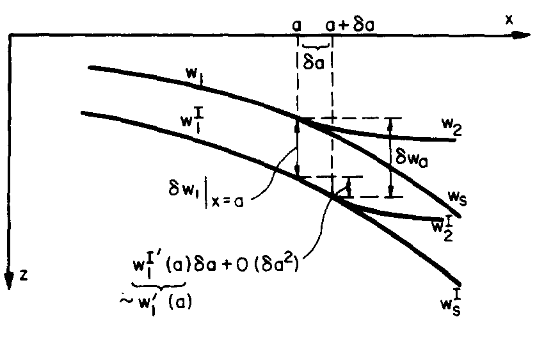

Fig. 5.

$$[F_1 - F_2 - F_s]_{x=a} = 0 \quad \text{(transversality condition)} \tag{3.19}$$

the BC's at $x = l$ are

$$\left[\frac{\partial F_2}{\partial w_2''}\right]_{x=l} = 0 \tag{3.20}$$

$$\left[\left(\frac{\partial F_2}{\partial w_2''}\right)'\right]_{x=l} = 0 \tag{3.21}$$

and the BC at $x = L$ is

$$w_s(L) = 0 \quad \text{or} \quad \left[\frac{\partial F_s}{\partial w_s'}\right]_{x=L} = 0. \tag{3.22}$$

The conditions (3.12)–(3.22) are the eleven equations needed for the determination of the ten integration constants of the three DE's in (3.8) and the unknown a.

For the specific problem under consideration, the Euler equations in (3.8) become, noting the relations in (3.2),

$$\left.\begin{aligned} (EIw_1'')'' - (Gw_1')' + kw_1 = q \quad & 0 \leqq x \leqq a \\ (EIw_2'')'' = q \quad & a \leqq x \leqq l \\ (Gw_s')' - kw_s = 0 \quad & a \leqq x \leqq L \end{aligned}\right\} \tag{3.8'}$$

the BC's at $x = 0$ become

$$w_1'(a) = 0 \tag{3.12'}$$

$$[-(EIw_1'')']_{x=0} = \frac{P}{2} \tag{3.13'}$$

the matching conditions at $x = a$ are

$$w_1(a) = w_2(a) \tag{3.14}$$

$$w_1(a) = w_s(a) \tag{3.15}$$

$$w_1'(a) = w_2'(a) \tag{3.16}$$

and

$$w_1''(a) = w_2''(a) \tag{3.17'}$$

$$[-(EIw_1'')' + (EIw_2'')' + G(w_1' - w_s')]_{x=a} = 0 \tag{3.18'}$$

$$w_1'(a) = w_s'(a) \tag{3.19'}$$

the BC's at $x = l$ are

$$w_2''(l) = 0 \tag{3.20'}$$

$$[(EIw_2'')']_{x=l} = 0 \tag{3.21'}$$

and the BC at $x = L$ is

$$w_s(L) = 0 \quad \text{or} \quad w_s'(L) = 0. \tag{3.22'}$$

It is of interest to note that according to matching condition (3.19′), which is the transversality condition, the slope of the shear layer at point $x = a$ is continuous, unlike at $x = l$ of Example I where the slope is discontinuous. Note that due to eqn (3.19′) the third term in condition (3.18′) vanishes. Thus, *at the separation point of beam and shear layer there is no concentrated reaction force.*

The above problem (Fig. 4) for $q = q_0 = \text{const.}$ was recently analyzed by Chernigovskaya, who formulated the problem intuitively. Instead of the matching conditions (3.17′)–(3.19′), Cheringovskaya used ([5], p. 124)

$$-EIw_1''(a) = -\frac{q_0(l-a)^2}{2} \tag{3.23}$$

$$-EIw_1'''(a) = q_0(l-a) \tag{3.24}$$

$$-Gw_1''(a) + kw_1(a) = 0. \tag{3.25}$$

Condition (3.23) which represents moment equilibrium at $x = a$, is equivalent to eqn (3.17′). Condition (3.24) is equivalent to eqn (3.18′), since the third term in (3.18′) is equal to zero. Note, however, that Chernigkovskaya's footnote (p. 124) stating that the formally exact condition would have to include a concentrated reaction force at $x = a$ is not correct since, as shown above, at the point of separation of beam and shear layer a concentrated reaction force does not exist. The third condition, (3.25), is not correct. It states that the distributed reaction pressure at $x = a$ is equal to zero. The correct equation is the transversality condition (3.19′) which states that at $x = a$ the slope of the shear layer is continuous.

The same comments apply to Chernigovskaya's formulation of a circular plate on a Pasternak base ([5], p. 135).

CONCLUSIONS

On three examples it is shown how to properly formulate problems in structural mechanics for which the intuitive approach led to incorrect formulations of boundary and matching conditions. The presented derivations are based on the principle of stationary total potential energy and the methods of variational calculus for variable end points.

The obtained results show that: (1) at the "free" end of a beam which is continuously attached to a Pasternak or "generalized" base, there usually exists a concentrated reaction force that enters the BC's. It should be noted that this is also valid when the Pasternak foundation does not extend beyond the beam end, and (2) for the beam which only rests on a Pasternak foundation (and thus can lift off the base over certain intervals), at the separation point of beam and base the slope of the shear layer is continuous and thus there is no concentrated reaction force.

In conclusion, the great utility of the method of variational calculus for variable matching points should be pointed out for the formulation of matching conditions of adjoining regions that are governed by different differential equations.

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